COUNTEREXAMPLE TO THE TANGHERLINI ARGUMENT*

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Abstract: Tangherlini has stated a set of postulates which lead to Schwarzschild's metric without the use of field equations. These postulates are shown to be inconsistent when applied to the parallel vacuum field.

By an odd coincidence, Schiff's eight-year-old paper,[1] and some others, purporting to obtain from the equivalence principle the general-relativistic bending of light (or, what amounts to the same thing, the coefficients of Schwarzschild's metric to first order) were recently criticized in these pages in two separate articles, one by Sacks and Ball,[2] the other by me.[3] In their conclusion Sacks and Ball refer with apparent concurrence to a set of postulates of Tangherlini[4] which undoubtably does yield the Schwarzschild metric to first order without the use of field equations. It is my purpose here to show that these postulates work for the Schwarzschild metric only accidentally, since they are inconsistent in another equally simply situation. I use the same counterexample as for the Lenz-Schiff argument, namely the static parallel vacuum field with metric

$$ds^2 = X^2 dT^2 - dX^2 - dY^2 - dZ^2 , \quad (1)$$

which is related to the usual Minkowski metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad (2)$$

by the coordinate transformation

$$x = X\cosh T \ , \ y = Y \ , \ z = Z, \ t = X\sinh T \quad . \tag{3}$$

Tangherlini's chief postulate is that in a static field the "acceleration" of a particle moving geodetically in the direction of the field should depend on position only. However, he defines acceleration in a somewhat unusual way as the second derivative of the space variable with respect to proper time. (With coordinate time, instead, the postulate is false in the case of Schwarzschild space.) As shown by Tangherlini in the spherically symmetric case, the following is true also in the "linear" case, i. e., for static metrics without cross terms whose coefficients depend on one spatial coordinate only (say x^1): for geodetic motion in the x^1 direction d^2x^1/ds^2 is velocity-independent if and only if $g_{oo}g_{11}$ is constant, which can be normalized by a simple change of time scale to

$$g_{oo}g_{11} = -1 \ . \tag{4}$$

(See Appendix below.) The unique coordinate transformation which casts the metric (1) into the required form is

$$X^2 = 2\xi \ , \tag{5}$$

whereupon it becomes

$$ds^2 = 2\xi dT^2 - (1/2\xi)d\xi^2 - dY^2 - dZ^2 \ . \tag{6}$$

Since a particle moving freely (i. e., geodetically) along the x axis of Minkowski space satisfies the equations

$$dx/dt = v = \text{constant}, \quad ds^2 = (1 - v^2)dt^2 , \tag{7}$$

and since Eqs. (3) and (5) imply

$$2\xi = X^2 = x^2 - t^2 ,$$

it follows that, for a particle moving geodetically in the ξ direction,

$$d^2\xi/dt^2 = v^2 - 1, \quad d^2\xi/ds^2 = -1 . \tag{8}$$

Tangherlini's acceleration for a free particle in the parallel field (6) is therefore constant, as anticipated. But, by another postulate, he equates his with Newton's acceleration in the field. It is now that an inconsistency appears: the actual force (or proper acceleration) felt by an observer at rest in the parallel field is well known to be $1/X$, i. e., $1/(2\xi)^{1/2}$, and not constant. In fact, it becomes infinite at $\xi = 0$.

It could perhaps be argued that the "Newtonian force" should be calculated by applying Gauss's theorem to a bundle of lines of force in the familiar way, which would indeed yield a constant force in the parallel field and thus save Tangherlini's postulates still in this case. But I would regard this as rather far-fetched.

Appendix

We here derive the condition (4). Consider a metric

$$ds^2 = -A(dx^1)^2 - B(dx^2)^2 - C(dx^3)^2 + D(dx^4)^2 ,$$

with A, B, C, D functions of x^1 only. The x^4 geodesic equation is

$$(x^4)'' + \Gamma^4_{ij}(x^i)'(x^j)' = 0 , \tag{9}$$

where a prime denotes d/ds. Consulting a list of Γ's (e. g., Dingle's list, reproduced in R. C. Tolman, Relativity, Thermodynamics, and Cosmology, Oxford University Press, Oxford, England, 1934, page 254), we find that the only nonzero Γ^4_{ij} is $\Gamma^4_{14} = (dD/dx^1)/2D$. Writing t for x^4, Eq. (9) thus becomes

$$t'' + (1/D)D't' = 0 ,$$

or, on integrating,

$$t' = k/D , \tag{10}$$

where k is evidently velocity-dependent, being the only disposable constant. If we write x for x^1 and substitute Eq. (10) into the metric with $x^2 = x^3 = 0$, we get

$$1 = -Ax'^2 + k^2/D , \tag{11}$$

whence

$$2x'' = \frac{d}{dx}\left(\frac{k^2}{AD} - \frac{1}{A}\right) .$$

We see that x'' will be velocity-independent if and only if AD is constant, which is equivalent to Eq. (4). That Eqs. (10) and (11) (supplemented by $x^2 = x^3 = 0$) indeed specify a geodesic becomes evident on checking the x^2 and x^3 geodesic equations: since $\Gamma^\mu_{11} = \Gamma^\mu_{44} = \Gamma^\mu_{14} = 0$ ($\mu = 2, 3$), they are satisfied ; and three geodesic equations suffice to determine a geodesic.

References

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4. F. R. Tangherlini, Nuovo Cimento 25, 1081 (1962).